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FINAL TECHNICAL REPORT

NASA GRANT NGL 33-010-070

FINITE ELEMENT SHELL INSTABILITY ANALYSIS

November, 1975

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# I. INTRODUCTION

This final report on NASA Grant NGL 33-010-070, "Finite Element Shell Instability Analysis" summarizes the work accomplished under the grant from its inception in November, 1968 until 1 July, 1975, the date of termination of the grant. The overall objectives of this work were to formulate procedures and an associated computer program for finite element thin shell instability analysis. In order to meet these objectives, the research was divided into the following component aspects:

(a) formulation of basic element relationships

(b) construction of solution algorithms on both the conceptual and algorithmic levels

(c) conduct of numerical analyses to verify the accuracy and efficiency of the theory and related programs.

These aspects and the results achieved therein are described in subsequent sections of this report. First, however, a section is devoted to a brief description of the background of finite element thin shell instability analysis. The body of the report concludes with remarks concerning theoretical directions and numerical schemes, identified in the course of the grant research, which could not be pursued in detail in conjunction with the grant but which hold much promise for improvements in finite element thin shell instability analysis.

NASA Contractor Reports produced under this grant, as well as more informal reports and papers published or presented with support of the grant (or deriving perspectives therefrom) are represented in the References of this report.

## II. BACKGROUND

The problem of finite element thin shell instability analysis represents the confluence of three major technological disciplines, each of which has been the subject of intensive and widespread study in past years. These disciplines are thin shell theory, structural instability theory, and finite element analysis. Other disciplines have been also involved, such as numerical methods (eigenvalue calculation, numerical integration) and computer programming, but this is usually regarded as either subsidiary or peripheral to those delineated above.

In thin shell theory the clarification of appropriate strain-displacement relationships has been a relatively recent development (since 1960). Certain formulations have been known to be deficient in that they give rise to strain under rigid body motion. Other formulations do not possess such shortcomings and differ from one another only with respect to terms which are unimportant in comparison to the assumptions made in formulating a thin shell theory. These have been identified in papers by Koiter (Ref. 1) and by Sanders (Ref. 2) and characterized as "consistent" formulations. The present work has basically adopted the consistent formulation due to Koiter (Ref. 1). Koiter's equations apply only to linear analysis; shell instability analysis requires the inclusion of terms that represent geometrically nonlinear behavior. The terms formulated by Mushtari and Galimov (Ref. 3) are appropriate to the latter and have been adopted in the work described herein.

Structural instability theory has sustained a similar growth of understanding. The parameters to be solved for differ from one

physical circumstance to another and they do not pertain exclusively to a tracing of the load-displacement response along a continuous curve, starting from the linear regime at small loads into significant nonlinearities at higher loads. Rather, the diverse situations portrayed in Figure 1 are encountered. This figure examines the load-displacement behavior of different types of structural action, tracing in each case the response of a representative degree-of-freedom.

The solid line of Figure 1 applies to "perfect" structures and represents the case in which the structure first displaces along the path defined by OAB (the fundamental path) and bifurcates (or branches) at the point A. The postbuckling path may rise (path AC), as in the case of compressed perfectly-flat plates, or it may descend (path AD), as in the case of compressed, perfect, thin-walled cylinders with appropriate support conditions.

When the structure possesses fabricational imperfections, the load-displacement behavior in the presence of destabilizing phenomena follows the paths indicated by dotted lines (Figures 1a and 1b). Structures with a rising post-buckling path as in Figure 1a (e.g., flat plates) will have strength exceeding the bifurcation load. The strength of an imperfect structure with a descending post-buckling path in the perfect state, represented by Figure 1b, will not achieve strengths as high as the bifurcation load unless the load-displacement path again rises at larger displacements. Such structures, under the appropriate load condition, are therefore imperfection sensitive and the maximum load attained (Point E) is termed the limit point.

A non-bifurcating load-displacement behavior may also occur for a structure assumed to be devoid of imperfections and may take the form similar in shape to the curve OE (Figure 1b) of the imperfection-sensitive structure. Again, a limit point is encountered and the buckling phenomenon is of the snap-through type.

Four component areas of structural instability theory may therefore be perceived from the foregoing: (1) general nonlinear prebuckling analysis, (2) the calculation of limit points, (3) calculation of bifurcation (or "branching") points, and (4) determination of the load-displacement response along a post-buckling path.

Surveys of the finite element method in structural instability theory have appeared at various times since 1969, when Martin (Ref. 4) presented a view of the topic. More recent surveys have appeared in Refs. 5-8.

In regard to finite element analysis it is fair to say that no acceptable curved thin shell element was available at the start of the current research. The preferable formulation would permit accurate analyses at reasonable cost and be reliable in the analysis of the widest range of shell forms and loadings. The desirability of meeting all relevant conditions on a finite element formulation-- but especially those on zero-strain under rigid body motion, inter-element continuity of displacement, and the representation of constant strain--is widely accepted. All of these cannot be achieved at reasonable computational cost, however, and the intervening years have seen extensive studies of the tradeoffs that can be made. A recent summary of the state-of-the-art in finite element thin shell analysis has been given in Ref. 9.

## III. ELEMENT FORMULATION

Major components of the efforts on this grant were devoted to the formulation of the finite elements needed for instability analysis of general curved shells. The basic element is an arbitrary triangular shell element (Fig. 2), although formulations were accomplished for an arbitrary quadrilateral element (Fig. 3) and a stiffening element (Fig. 4).

As noted in the prior section, the entire field of finite elements for shell analysis was in its infancy at the start of the work on this grant. Although linear shell analysis is well in hand, the amount of work done with respect to elastic instability analysis in this mode continues to be quite limited.

The usual conditions on a finite element formulation (compatibility, constant strain, rigid body motion) have been approached, if not met exactly, through formulation of the triangular element on the basis of the Koiter (Ref. 1) "consistent" shell theory and cubic polynomial descriptions of the three displacement components. This formulation does not have the required interelement continuity of displacement, so continuity is "restored" by writing constraint conditions which are appended to the analysis with use of the Lagrange multiplier method.

The formulation of the above element, which is done in terms of orthogonal curvilinear coordinates and with nonzero Gaussian curvature, is described in detail for linear analysis in CR-2482 (Ref. 10). This report includes extensive numerical solution data which demonstrates that the approximations made with respect to the condition on zero-strain under rigid body motion are satisfactory for all of the conventional problems studied as well as for

the particular pathological cases studied. The latter were intended to emphasize any deficiencies which might arise on account of the failure to meet exactly the condition on zero-strain under rigid body motion.

The extension of the triangular thin shell element to account for geometrically nonlinear behavior is detailed in CR-2483 (Ref. 11). Here, the shell equations are extended to the description of nonlinear behavior with terms derived by Mushtari and Galimov (Ref. 3). The finite element representation of these terms was accomplished with the same displacement assumptions as for the linear terms ("consistent" formulation), as well as with simpler assumptions ("inconsistent" formulation). Also, a combination of four triangles was made in formation of a quadrilateral element.

It should be noted that evaluation of the strain energy integrals that give the element stiffness coefficients was performed by means of numerical integration, using Gauss-Radau quadrature. Numerical studies of certain problems, particularly those featuring spherical caps, disclosed that the rapid variation of geometric parameters in the vicinity of the crown could not be adequately handled by linear variation of these parameters within the element. By implementation of a capability to evaluate the geometric parameters at each numerical integration point it was possible to obtain highly accurate results for the problems in question. This more recent work is described in Ref. 12.

## IV.  SOLUTION ALGORITHMS - CONCEPTUAL

It was noted in Section II of this report that there are four divisions of the general topic of shell instability analysis: (1) nonlinear prebuckling, (2) bifurcation, (3) limit point, and (4) postbuckling.

In the present work, in nonlinear prebuckling analysis, the approach taken was the incremental-iterative method.  The load path is first divided into a number of increments.  The tangent stiffness is formed for a typical interval (see Fig. 5) and the linearized analysis is performed.  Then, the calculated displacements are substituted into the equilibrium equations to yield a "residual", which represents the error due to the linearization of the analysis.  The residual is employed in an "initial-force" corrective analysis for the interval; this step does not require re-formation of the stiffness in the interval.  After the initial force correction is made the analysis proceeds to the next interval.

The calculation of bifurcation buckling, for most of the grant, was performed by interpolation of the determinant of the nonlinear prebuckling analysis stiffness equations through the zero value.  This is based on the fundamental condition that at buckling the stiffness matrix is singular, i.e., has a zero value for its determinant.  Subsequent experience showed this procedure to be unreliable, since the plot of determinant versus load might be erratic.  Thus, in the latter part of this research the Sturm-sequence algorithm of Gupta (Ref. 13) was implemented and adopted.  This algorithm is foolproof in the definition of the lowest eigenvalue.

To trace the load-deflection curve up to and past the limit point special precautionary measures have to be taken. At the limit point the determinant of the stiffness matrix is zero and, consequently, the system of equations does not possess a unique solution. The deficiency is of rank one so to overcome this one constraint must be imposed on the equations. The present constraint consists of the assignment of a prescribed value to one of the displacement degrees-of-freedom.

The procedure is as follows. First, a representative degree-of-freedom is identified. The central displacement of a plate or shell is an appropriate choice, the selection being input directly by the user. Three load vectors are dealt with:

(1) No loading, except for the reaction at the prescribed displacement: $\{P_1\}$

(2) The incremental load vector, with zero displacement at the representative freedom: $\{P_2\}$

(3) The out of balance load vector calculated on the basis of the displacements at the last load level, with zero displacement at the representative freedom: $\{P_3\}$

These three load cases are shown in Fig. 6 for an arch. The reactions at the constrained freedom are denoted by $R_1$, $R_2$ and $R_3$, respectively. The desired combination of displacements is made up of two separate combinations thus

$$\{\Delta_1\} - \frac{R_1}{R_2} \{\Delta_2\}$$

and

$$\frac{R_3}{R_1} \{\Delta_1\} + \{\Delta_3\}$$

The first equation gives the necessary combination in the absence of out of balance forces. This would be sufficient in the case of the regular incremental approach. In that case $R_1$ would be zero at the limit point but $R_2$ would be nonzero and there would be no computational difficulty.

In postbuckling analysis it is necessary to distinguish between limit point and bifurcation buckling. Limit point post-buckling can be handled by the procedures described above, i.e., by incrementing displacement rather than load. In bifurcational buckling it is necessary to extract the prebuckling path. This can be done by perturbation procedures, developed initially by Koiter (Ref. 14) for classical stability analysis and subsequently formulated by Thompson, et al (Ref. 15) for multiple degree-of-freedom (discrete coordinate systems). A rather complete approach to finite element postbuckling analysis, based on perturbation procedures, was developed under this grant and is described in Refs. 16-18.

## V. SOLUTION ALGORITHMS - NUMERICAL

Efforts were devoted, during the full span of the grant, to the development of a computer program for finite element shell instability analysis ("FESIA"). This program, coded in Fortran IV, is documented in Ref. 19, has been operational at both Cornell University and NASA Langley Research Center. Ref. 19 comprises a Level 3 program documentation as defined by NASA (Ref. 20). This means that theoretical background is outlined, operational, input, and output instructions are detailed and exemplified, and a flow chart of macro-operations is given. In view of the scope and content of Ref. 19 the nature of the program is only sketched in the following.

The cornerstone of the program is the triangular thin shell element described in Section III. The element may have orthotropic material properties. The geometric description of the element permits non-zero Gaussian curvature referenced to a system of orthogonal curvilinear coordinates. Program options permit simplified geometric description when common shapes (e.g., circular cylinders) are to be analyzed. The definition of support conditions is also simplified via the inclusion of familiar circumstances, such as fixed support. Point loads, distributed loads, and pre-assigned joint displacements can be specified. In the case of distributed loads a highly accurate description is possible, since the evaluation is performed at all numerical integration points within the element. It should be noted that a quadrilateral element can be specified which is formed of four triangular elements.

The program permits a general, geometrically nonlinear analysis based on a Lagrangian description of strain-displacement equations

and the incremental-iterative computational scheme. The load-displacement behavior can be carried beyond the limit point through use of displacement incrementation. Bifurcational buckling is determined through either an interpolation of the determinant or by use of Gupta's Sturm sequence algorithm (Ref. 13).

A variety of analyses performed with use of the FESIA program, intended for comparison with alternative solutions, is described in the next two sections.

## VI. VERIFYING ANALYSES

In any large-scale computational approach it is essential that the adequacy of the formulation and of the underlying approximations be verified through performance of comparisons with alternative theoretical solutions and test data. Availability of such information is limited for the case of elastic instability of general curved thin shells.

A rather extensive series of verifying analyses for the formulations developed under this grant has been reported in the relevant Contractor Reports and published papers.
These include the "standard" comparison problems of the cylindrical shell roof originally analyzed by Scordelis and Lo (Ref. 21), the pinched cylinder (Ref. 22), a hyperbolic paraboloid shell (Ref. 23), a torus (Ref. 24), and a pinched and pressurized sphere (Refs. 25 and 26). In all cases the solution accuracy and computational efficiency was found to be comparable, if not superior, to alternative formulations.

During the final year of the grant some comparison analyses were performed of two particular problems which have represented a long-standing challenge in the prediction of shell elastic instability. These were the torispherical shell under internal pressure and the hyperboloid under wind loading.

In 1959 Galletly (Ref. 27) called attention to the fact that when torispheres are used as the closures of internally-pressurized tanks, circumferential compressive stresses are introduced which may lead to an elastic instability mode of failure. The problem is quite complex due to the rapidly-varying state of stress and the localized nature of the buckle modes.

Figure 7 shows the finite element idealization used in analysis of this problem. The capability of accurately representing the rapidly changing geometric parameters near the crown was especially important. Figure 8 shows the level of agreement between an alternative solution for the prebuckling (linear) stresses (Ref. 24) and the finite element solution; these are seen to be in reasonable agreement. With respect to buckling, the bifurcation solution obtained in this study was 6.68 p.s.i., compared with an experimental value of 6.7 in the experiment of Adachi and Benicek (Ref. 28).

The torispherical head represented an especially difficult bifurcational analysis condition. The lowest positive eigenvalue corresponded to buckling under external pressure, a negative value. Also, the relationship between the load intensity and determinant was quite erratic, requiring use of the Sturm sequence algorithm (Ref. 13).

The hyperboloid under wind loading (Fig. 9) also represents a problem with rapidly varying (spatially) prebuckling stresses. Much attention has been given to this problem because of the failure of a number of cooling towers under wind load in England in the mid-1960's. It should be noted that a similar structural form is encountered in many aerospace applications (e.g., rocket engine thrust chambers).

A finite element idealization of the cooling tower is seen in Figure 9. Figure 10 gives a comparison of results obtained with the FESIA program (Ref. 12) and those due to Fonder (Ref. 26) and Cole, et al (Ref. 34). They are seen to be in close agreement with the latter. A comparison of buckling test data (Ref. 30)

and the present solution showed a difference of 25% (Ref. 12).
This is closer agreement than recorded in studies performed else-
where.

## VII. THEORY-TEST COMPARISON

One theory-test comparison accomplished under this grant,
results of which have design significance, involved the shear
buckling of reinforced, perforated plates (Figs. 11, 12). A
practical question which has ⋅⁻ ₊t⁻d for many years concerns the
amount of reinforcing material that is needed to compensate for
the material removed in making the hole. A number of plates of
different proportions of reinforcement were tested under support
of the Cornell Structural Engineering Department and analyses of
these conditions were performed by means of the FESIA program.
The theory-test comparisons were good and disclosed that the
amount of reinforcement needed to compensate for the hole is con-
siderably less than had heretofore been assumed. This work is
described in a NASA CR (Ref. 31).

## VIII. CONCLUDING REMARKS

Not all of the theoretical work performed was implemented in the FESIA program as of the conclusion of the grant, and various schemes have been identified which hold much promise for the improvement of the accuracy, scope and efficiency of the program. These include:

(1) Treatment of interelement displacement discontinuity, in the case of the triangular element, can be accomplished in a manner which eliminates the need for Lagrange multipliers. One such approach has been proposed by Kikuchi and Ando (Ref. 32).

(2) Self-contained assessment of whether a problem is in the class of bifurcation buckling or limit point and succeeding computation strategy based on the result. Necessary theoretical basis is outlined in Ref. 33, a paper written under support of this grant.

(3) Implementation of various "condensation" schemes, enabling global analysis for bifurcation and limit point behavior to be conducted for a reduced number of degrees-of-freedom.

(4) Expanded representation of various types of elements, with special attention given to stiffeners and the verification of stiffened shell analysis.

(5) Improved input-output facilities. Emphasis should be placed on computer graphics, with use of picture tube representations and including mesh generation capabilities.

Work on all of the above considerations, unsupported by grant funds, was in progress at the time of writing of this report and will be made available to NASA when complete, if so desired.

## REFERENCES

1. Koiter, W., "A Consistent First Approximation in the General Theory of Thin Elastic Shells", in The Theory of Thin Elastic Shells, W. Koiter, Ed., North-Holland Publ. Co., Amsterdam, 1960, pp. 12-33.

2. Sanders, J. L., "Nonlinear Theories for Thin Shells", Quart. Appl. Math., 21, 1963, pp. 21-36.

3. Mushtari, K. and Galimov, K., Nonlinear Theory of Elastic Shells, NASA TT-F62, 1962.

4. Martin, H. C., "Finite Element Formulation of Geometrically Nonlinear Problems". Recent Advances in Matrix Methods of Struct. Analysis and Design. R. H. Gallagher, et al, Ed., Univ. of Alabama Press, 1971, pp. 343-381.

5. Stricklin, J., Haisler, W. and Von Riesemann, W., "Evaluation of Solution Procedures for Material and/or Geometrically Nonlinear Structural Analysis", AIAA J., V. 11, No. 3, 1973, pp. 292-299.

6. Gallagher, R. H., "Geometrically Nonlinear Finite Element Analysis", Proc. of Specialty Conf. on the Finite Element Method in Civil Engrg., McGill Univ., June 1972.

7. Yamada, Y., "Incremental Formulation for Problems with Geometric and Material Nonlinearities", Advances in Computational Methods in Struct. Mech. and Design, J. T. Oden, et al, Ed., Univ. of Alabama Press, 1972, pp. 325-356.

8. Hartung, R. (Editor), Numerical Solution of Nonlinear Structural Problems, Publ. AMD-6, ASME, N.Y., 1973.

9. Gallagher, R. H., "Shell Elements", Ch. 5, Proc. of World Congress on Finite Element Methods in Structural Mech., Bournemouth, England, October 1975.

10. Thomas, G. R. and Gallagher, R. H., A Triangular Thin Shell Finite Element: Linear Analysis, NASA CR-2482, 1975.

11. Thomas, G. R. and Gallagher, R. H., A Triangular Thin Shell Finite Element: Nonlinear Analysis, NASA CR-2483, 1975.

12. Kanodia, V., Finite Element Elastic Instability Analysis of Deep Shells of Double Curvature, Ph.D. Thesis, Cornell University, Ithaca, N.Y., December 1975.

13. Gupta, K. K., "Eigenproblem Solution by a Combined Sturm Sequence and Inverse Iteration Technique", Int. J. Num. Meth. Engrg., V. 7, 1973, pp. 17-42.

14. Koiter, W. T., *On the Stability of Elastic Equilibrium*, Ph.D. Thesis, Delft, 1945. (Published as AFFDL-TR-70-25, U.S.A.F. Flight Dynamics Lab., Dayton, O.).

15. Thompson, J.M.T. and Hunt, G. W., *A General Theory of Elastic Stability*, J. Wiley Publ., London, 1973.

16. Mau, S. T. and Gallagher, R. H., *A Finite Element Procedure for Nonlinear Pre-Buckling and Post-Buckling Analysis*, NASA CR-1936, January 1972.

17. Mau, S. T. and Gallagher, R. H., *A Method of Limit Point Calculation in Finite Element Structural Analysis*, NASA CR-2115, September 1972.

18. Gallagher, R. H., Lien, S. and Mau, S. T., "A Procedure for Finite Element Plate and Shell Pre- and Post-Buckling Analysis", *Proc. 3rd Conf. on Matrix Methods in Struct. Mech.*, AFFDL-TR-71-180, 1973, pp. 857-880.

19. Anon., *Computer Program Documentation Guideline*, NASA Publ. NHB 2411.1, July 1971.

20. Mang, H., Kanodia, V. and Gallagher, R. H., *User's Manual: FESIA Program*, Dept. of Structural Engrg. Report, Cornell University, Ithaca, N.Y., December 1975.

21. Scordelis, A. and Lo, K., "Computer Analysis of Cylindrical Shells", *ACI J.*, V. 61, 1964, pp. 539-561.

22. Morris, A. J., "A Deficiency in Current Finite Elements for Thin Shell Applications", *Int. J. Solids Struct.*, V. 9, 1973, pp. 331-346.

23. Brebbia, C. and Connor, J., "Analysis of Geometrically Nonlinear Plates and Shells by the Finite Element Method", Proc. ASCE, *J. of the Engrg. Mech. Div.*, V. 95, No. EM2, 1969, pp. 463-483.

24. Khojasteh-Bakht, M., *Analysis of Elastic-Plastic Shells of Revolution under Axisymmetric Loading by the Finite Element Method*, Report UC SESM 67-8, Structural Engrg. Lab., Univ. of Calif., Berkeley, 1967.

25. Cowper, G. R., Lindberg, G. and Olson, M., "Comparison of Two High-Precision Triangular Finite Elements for Arbitrary Deep Shells", *3rd Conf. on Matrix Methods in Struct. Mech.*, AFFDL-TR-71-160, Dayton, O., 1973, pp. 277-304.

26. Fonder, G., *A Doubly Curved Quadrilateral Element for Thin Elastic Shells of Revolution*, Report UC SESM 72-1, Structural Engrg. Lab., Univ. of Calif., Berkeley, 1972.

27. Galletly, G. D., "Torispherical Shells - A Caution to Designers", J. Engrg. for Industry, Trans. ASME, V. 81, 1959, pp. 51-62.

28. Adachi, J. and Benicek, M., "Buckling of Torispherical Shells Under Internal Pressure", Experimental Mechanics, V. 4, August 1964, pp. 217-222.

29. Albasiny, E. and Martin, D., "Bending and Membrane Equilibrium in Cooling Towers", Proc. ASCE, J. Eng. Mech. Div., V. 95, No. EM3, June 1967, pp. 1-17.

30. Der, T. J. and Fidler, R., "A Model Study of the Buckling Behavior of Hyperbolic Shells", Proc. Inst. C. E., (Gr. Brit.), V. 41, September 1968, pp. 105-118.

31. Grosskurth, J., White, R., Gallagher, R. and Thomas, G., Shear Buckling of Perforated Plates, NASA CR, 1975.

32. Kikuchi, F. and Ando, Y., "Some Finite Element Solutions for Plate Bending Problems by Simplified Hybrid Displacement Method", Nucl. Engrg. Des., V. 22, 1972, pp. 155-173.

33. Gallagher, R. H., "Perturbation Procedures in Nonlinear Finite Element Structural Analysis", Int. Conf. on Computational Methods in Nonlinear Mechanics Feature Lecture, published in Lecture Notes in Mathematics, No. 461, Springer, Berlin, 1974.

34. Cole, P., Abel, J. F. and Billington, D. P., "Buckling of Cooling Tower Shells: Bifurcation Results", Proc. ASCE, J. of Struct. Div., V. 101, No. ST6, June 1975, pp. 1205-1222.

Load

Bifurcation
Point

B

C

A

Displacement

O

a. Response of Actual
Structure Insensitive
to Imperfections

— Fundamental Path
--- Imperfect Structure
Response
...... Post Bifurcation Path

Load

Bifurcation
Point

A

E

D

Limit Point

Displacement

O

b. Limit Point Phenomenon
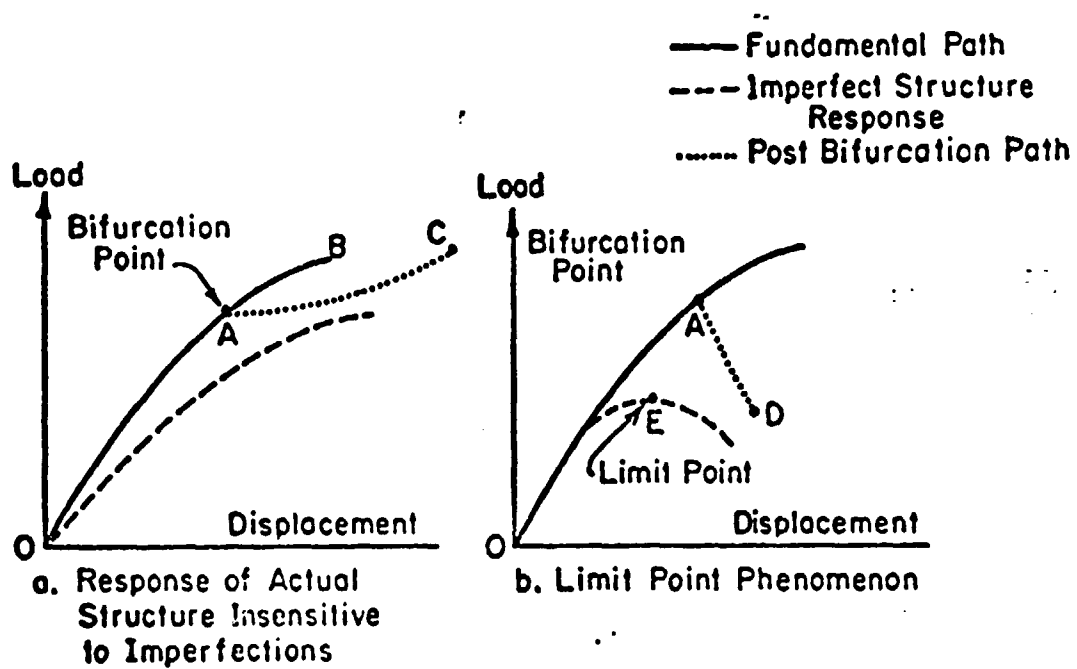
Figure 1.   Critical Load Phenomena
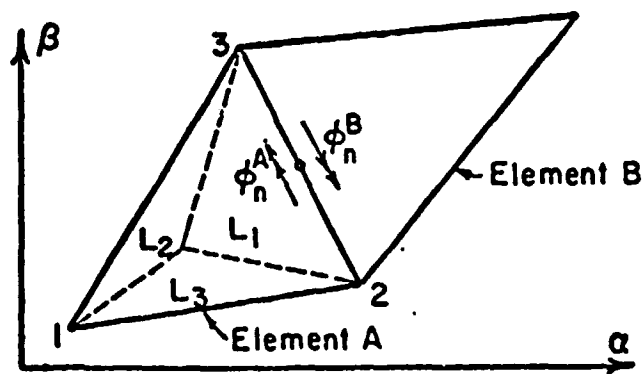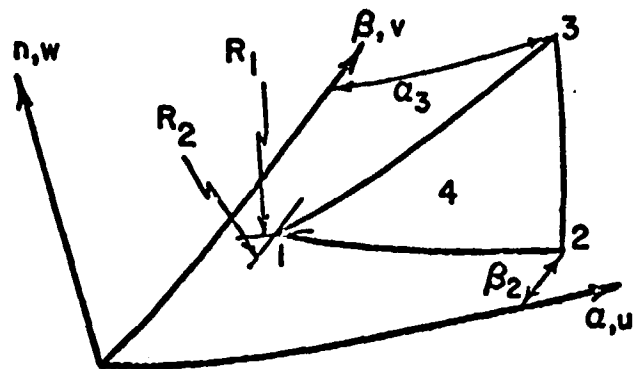
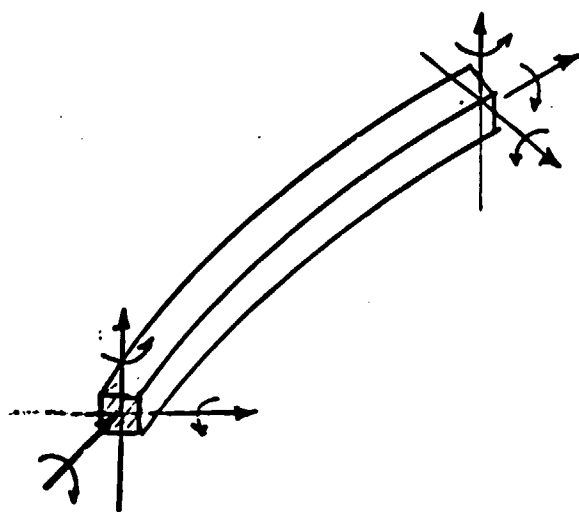Figure 2.  Triangular Thin Shell Element
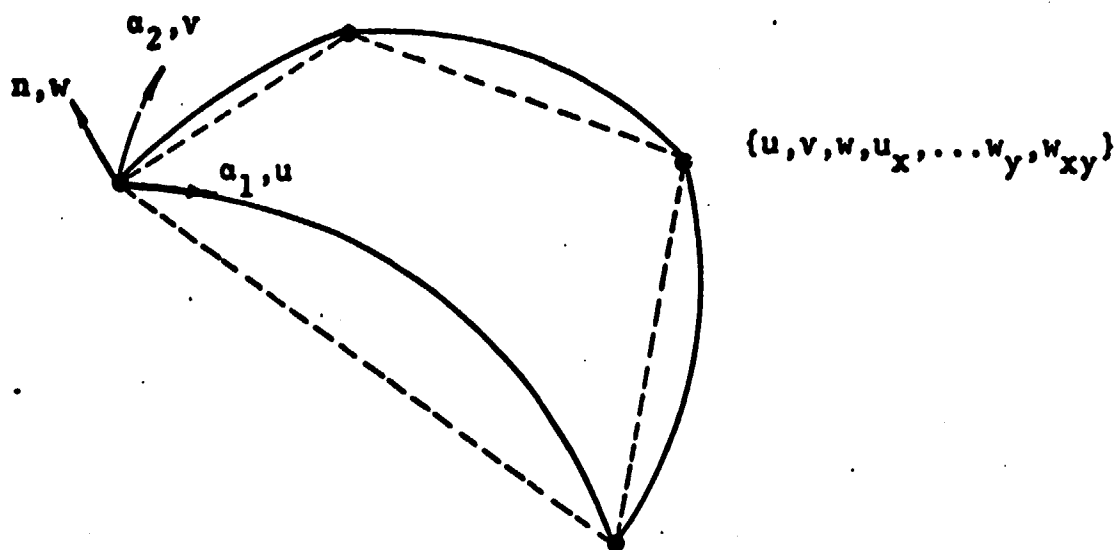
Figure 3.  Quadrilateral Element



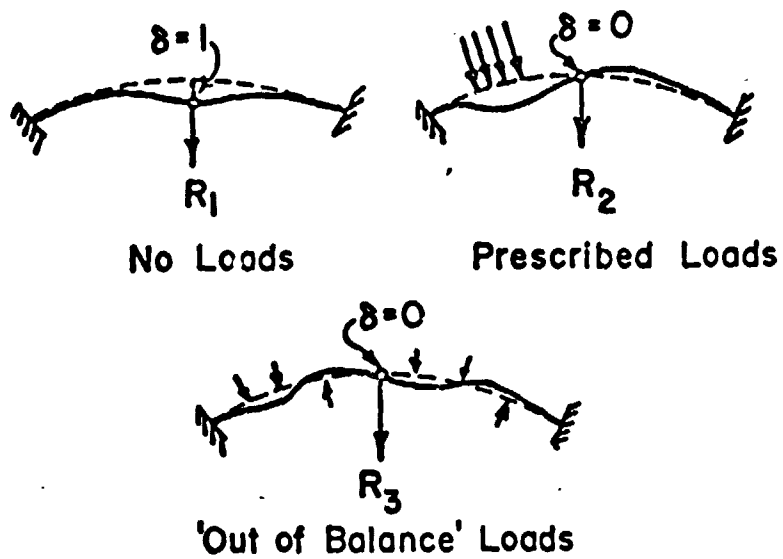Figure 4.  Curved Stiffener Element
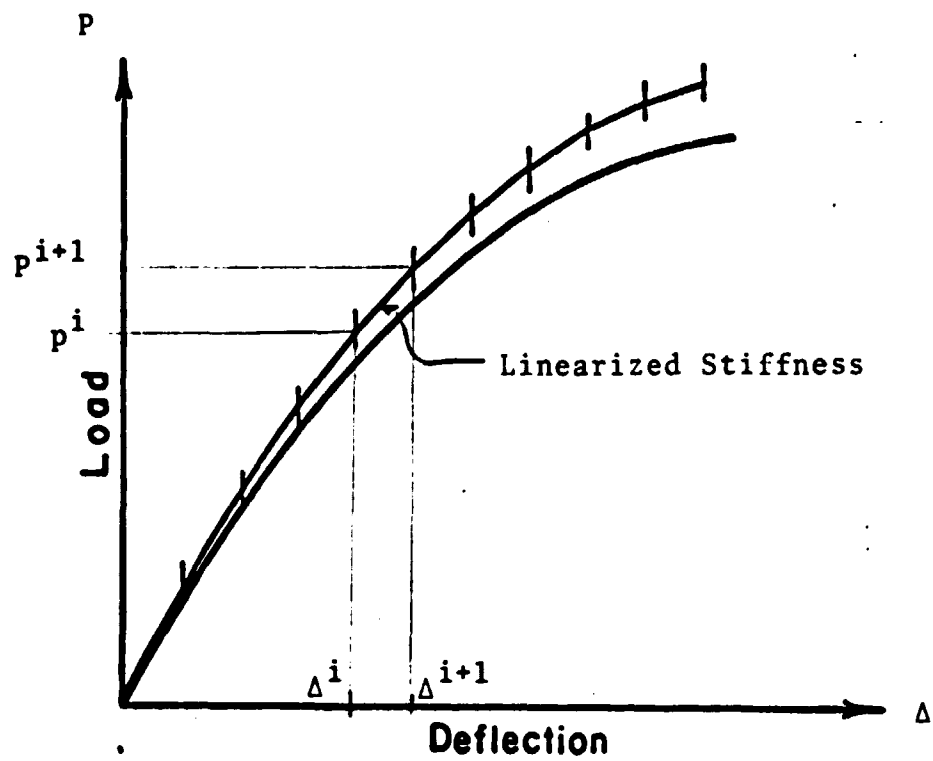
Figure 5.   Tangent Stiffness Analysis

Figure 6.   Method of Limit Point Calculation
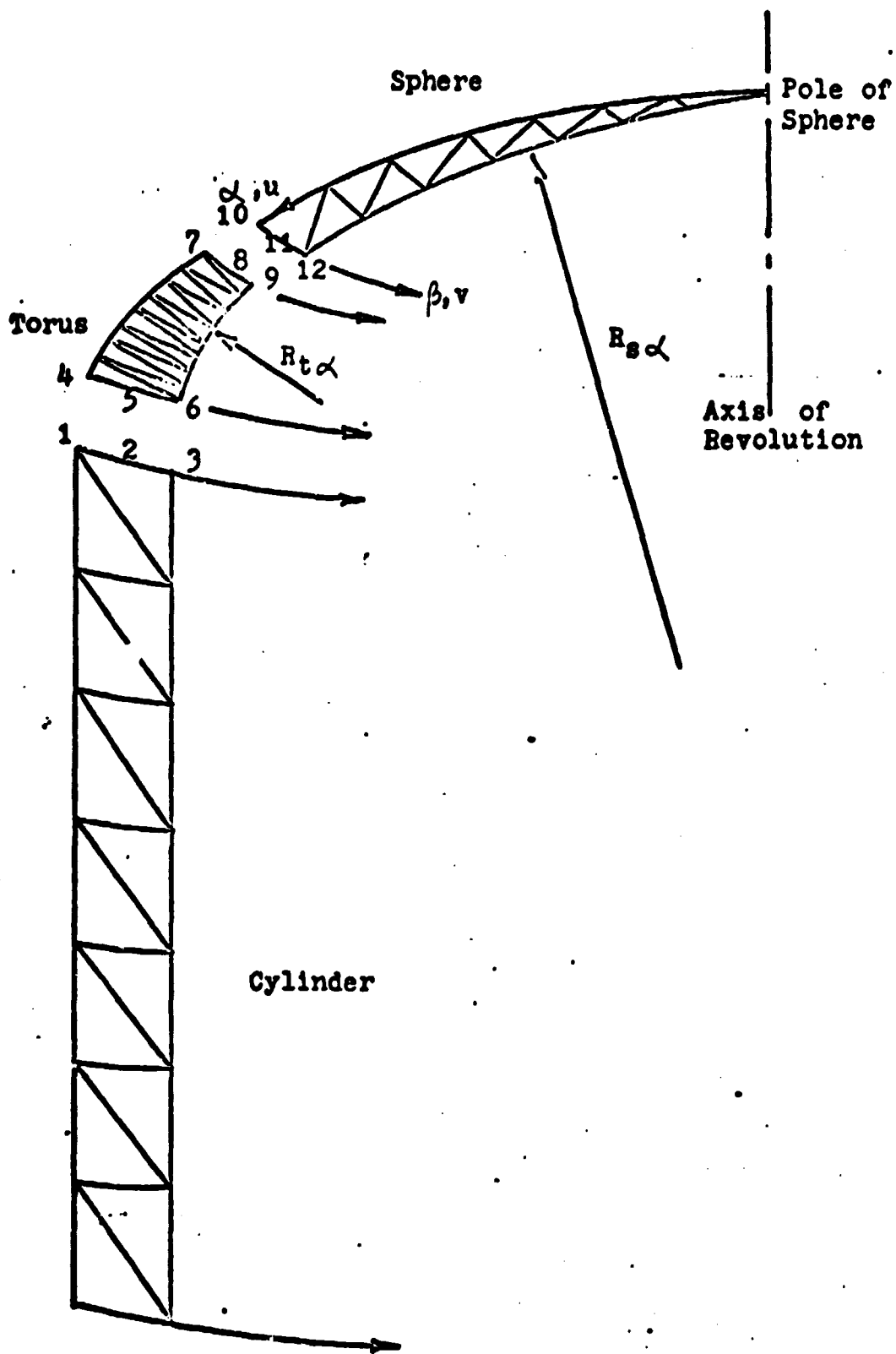
**Figure 7.** Finite element idealization of torispherical head pressure vessel and junction between component shells
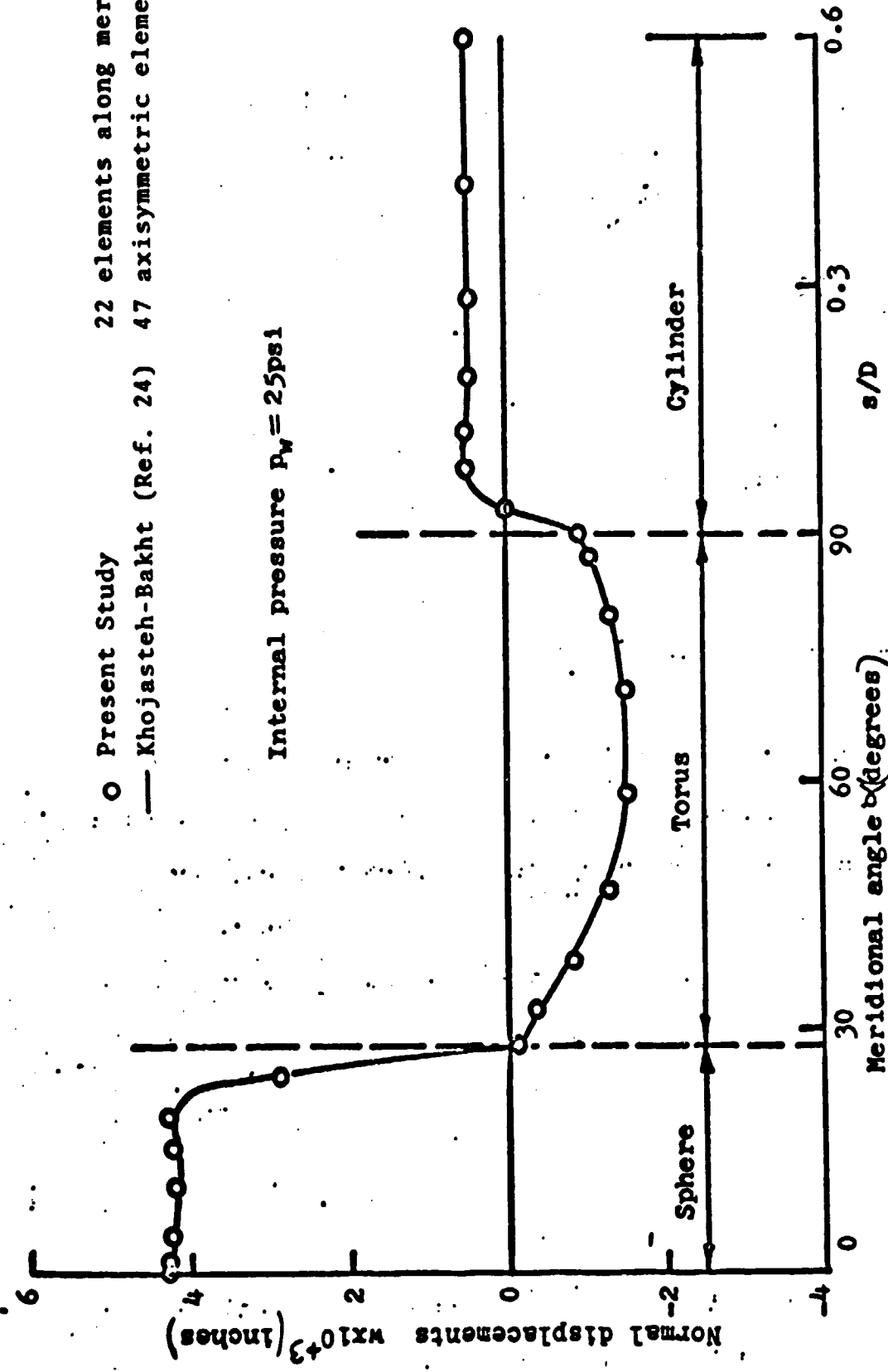
Figure 8. Normal displacements w for pressurized torispherical head vessel
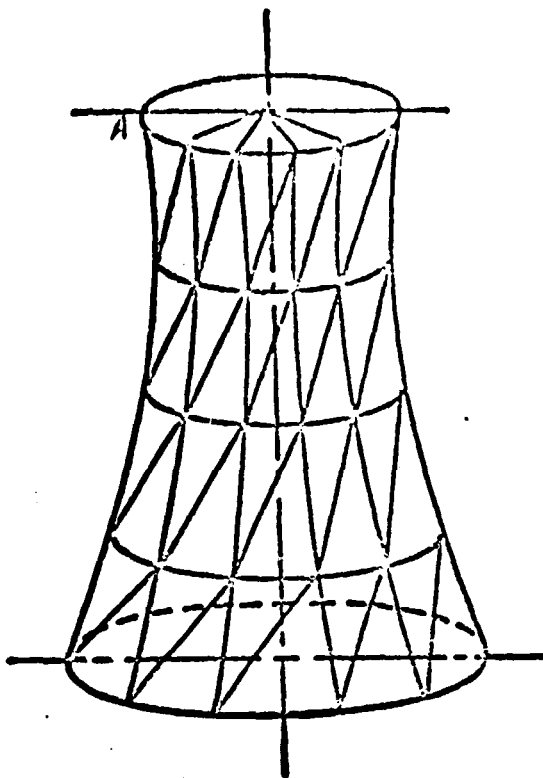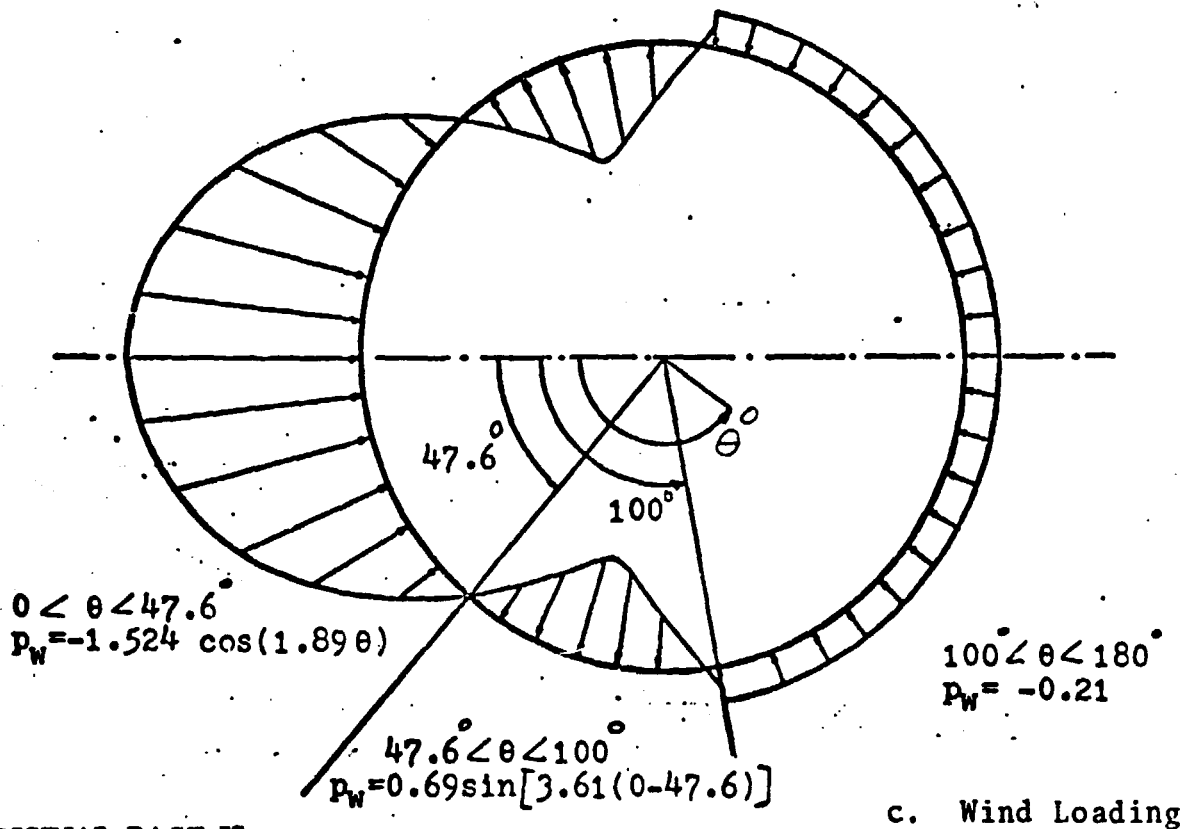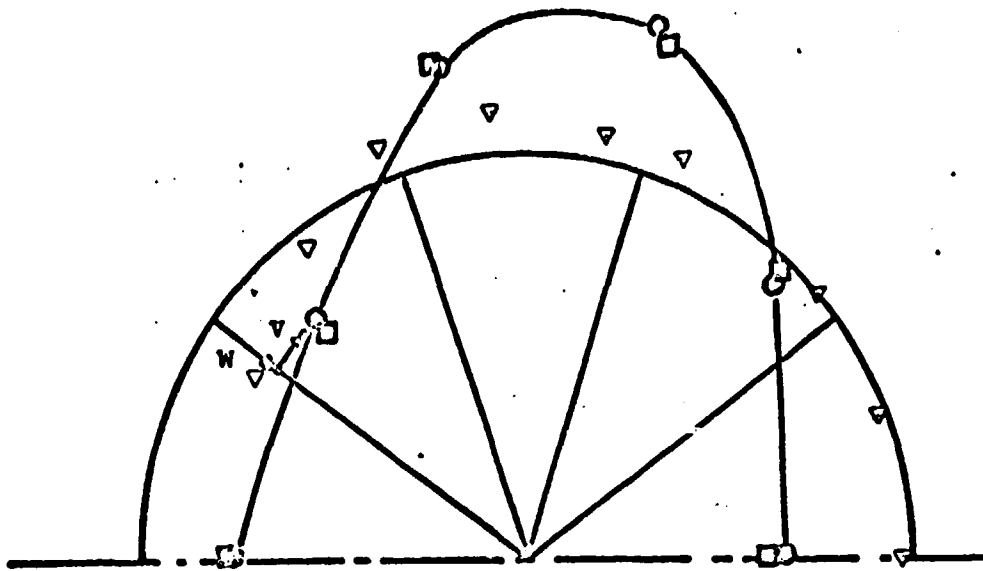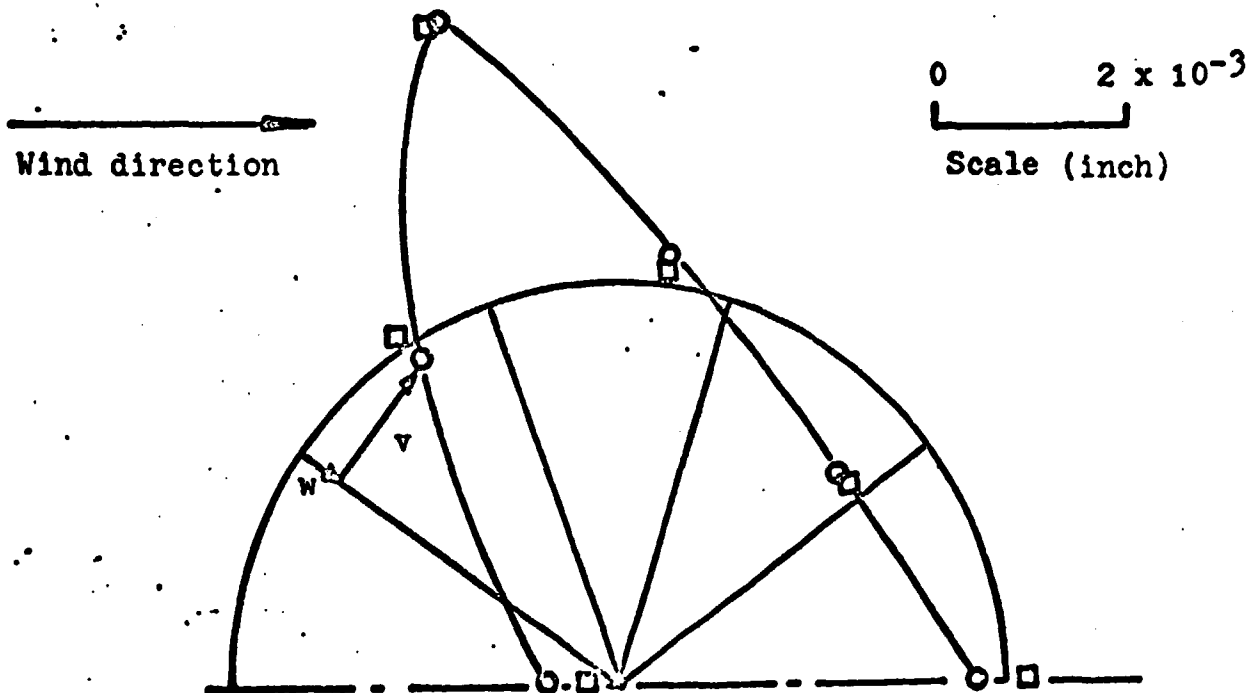
a. For wind loads
4 x 5 mesh
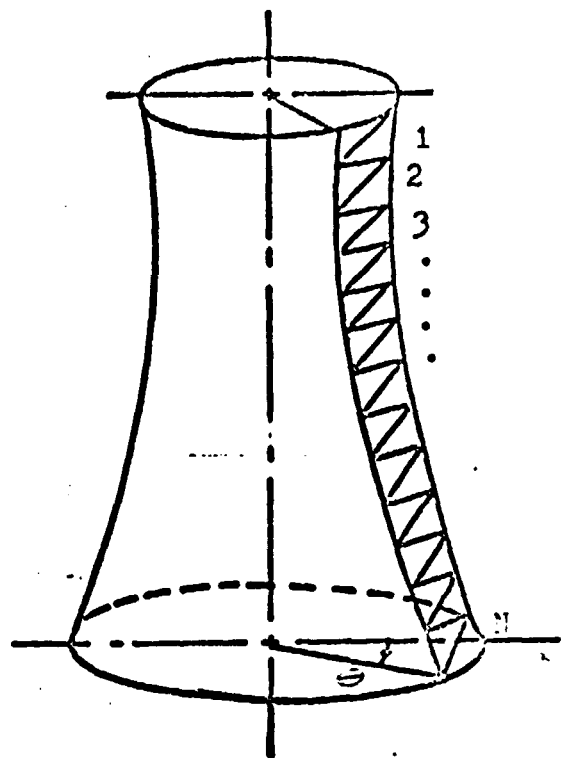
b. For dead load Nx1
mesh

Typical finite element idealizations
for hyperbolic cooling tower

$0 < \theta < 47.6°$
$p_W = -1.524 \cos(1.89\theta)$

$47.6° < \theta < 100°$
$p_W = 0.69 \sin[3.61(\theta - 47.6)]$

$100° < \theta < 180°$
$p_W = -0.21$

c. Wind Loading

Figure 9.

Horizontal section at $\phi = 0.0$

Wind direction

0          2 x 10$^{-3}$

Scale (inch)

Horizontal section at $\phi = 0.240$

—o— Present study  4 x 5 mesh
▼   Fonder (Ref. 26)  20 x 1 mesh
◻   Axisymmetric program of Reference 34; 31 elements

Figure 10.  Horizontal deformation of cooling tower
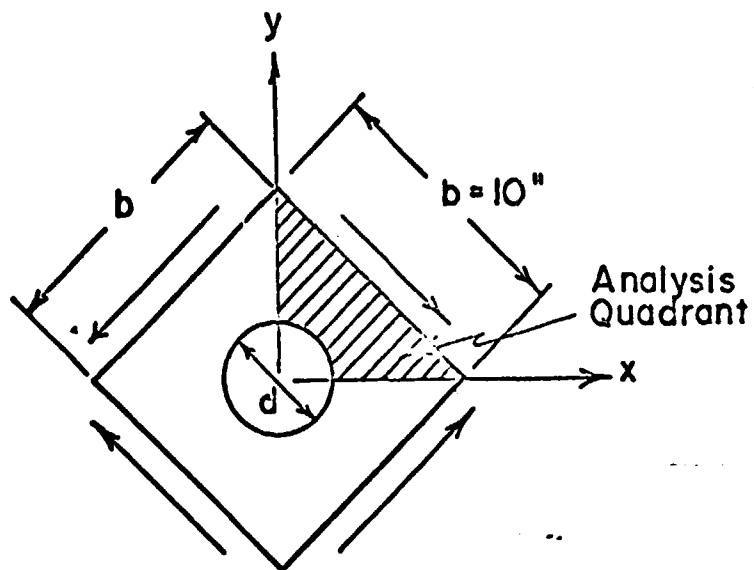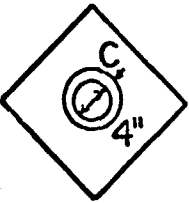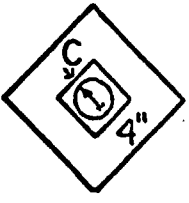under wind load at $\phi = 0.0$ and 0.240
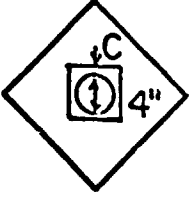
Figure 11. Unperforated Specimen

| Stiffener | Geometry | c, in | Vol. Stiffener / Vol. Removed Material |
|-----------|----------|-------|----------------------------------------|
| 1 | | 2 | 3.00 |
| 2 | | 1 | 2.50 |
| 3 | | 0.828 | 1.00 |
| 4 | | 0.506 | 1.00 |
| 5 | | 0.506 | 1.00 |

Figure 12. Reinforcement Patterns Around Perforations in Plates with d/b = 0.4